Problems for Group Contest

Problem 1. Prove that, for any integer $n \geq 1$, there exists a unique subgroup of $(\mathbb{Q}/\mathbb{Z}, +)$ of order n.

Problem 2. Let p be a prime, $\mathbb{F}_{p^k}$ the finite field of p^k elements, and ζ_p a primitive p-th root of unity in $\mathbb{C}$. For a positive integer d, define the algebraic integer

$$S_k(d) := \sum_{x \in \mathbb{F}_{p^k}} \zeta_p^{\mathrm{Tr}_k(x^d)} \in \mathbb{Z}[\zeta_p],$$

where Tr_k is the trace map from $\mathbb{F}_{p^k}$ to $\mathbb{F}_p$. Assume that d divides $\frac{p^k-1}{p-1}$. Prove that $S_k(d) \in \mathbb{Z}$.